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A Rigorous Study of Periodic Orbits by Means of a Computer

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We apply a modified version of the method of Sinai and Vul in order to study, by means of a computer, a closed orbit which appears in the five-mode model of bidimensional incompressible fluid on the torus.

KEY WORDS: Poincaré map; fixed points; fundamental matrix of solutions; pseudotrajectory.

INTRODUCTION

The study of closed orbits for systems of ordinary differential equations has got increasing importance in the last years because of the connection with the theory of turbolence as developed by Ruelle and Takens[(2)] and because of the Feigenbaum's conjecture[(3)] on the universal behavior of the sequence of bifurcations of periodic orbits.

The numerical studies of the Lorenz model have been very intensive and most of its properties have been clarified by Lanford.[(4)] Franceschini[(5)] and Franceschini and Boldreghini[(6)] also found periodic orbits, by means of a computer, respectively for the Lorenz model in a range of the parameters different from that explored by Lanford, and in the five-modes model for an incompressible bidimensional fluid.

In such a situation it is extremely interesting to have an exact theory which enables one to state under what conditions a certain periodic orbit found numerically is rigorously periodic. Sinai and Vul[(1)] have worked out

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a criterion to prove the existence and uniqueness of a fixed point of the Poincaré map in a neighborhood of a numerical fixed point by checking suitable conditions on the numerical trajectory.

In this work we use the same approach as in Ref. 1; the difference is that the estimate of the bound of the nonlinear part of the Poincaré map is obtained in a simpler way and that the bound obtained is smaller. Furthermore we take care of the round-off errors of the computer using the interval arithmetic.

All this machinery is applied for analyzing a periodic orbit found numerically in Ref. 6, i.e., in a system of ordinary differential equations corresponding to the five-dimensional truncation of the Navier–Stokes equations for an incompressible bidimensional fluid.

Section 1 is concerned with the definitions, the notations, and the main criterion. In Section 2 there are lemmas useful for evaluating the nonlinear part of the Poincaré map. The linear part of the Poincaré map is estimated as in Ref. 1. In Section 3 we deal with the numerical method and we give an analysis of the error connected with it.

The round-off error of the finite precision floating point arithmetic of the computer is controlled by using the method of interval arithmetic.[7] In Section 4 we present the results obtained in the case of the periodic orbit found in Ref. 6.

1. DEFINITIONS

We shall use the same symbols and notations as in Ref. 1.

Let $X = \{x_i, i = 1, \ldots, d\}$ be a vector in the space $\mathbb{R}^d$ with the scalar product $(X, Y) = \sum_{i=1}^{d} x_i y_i$.

We consider the differential equation in $\mathbb{R}^d$

$$\dot{X} = F(X) \tag{1.1}$$

where $F = \{f_i, i = 1, \ldots, d\} \in \mathbb{R}^d$ is defined by the equality

$$f_i(X) = (G^i, X) + (B^i X, X) + C^i, \qquad i = 1, \ldots, d$$

and G^i, $i = 1, \ldots, d$, is a constant vector B^i, $i = 1, \ldots, d$, is a constant $d \times d$ matrix, C^i, $i = 1, \ldots, d$, is a constant.

We denote with S_t the one-parameter group of shifts along the trajectories generated by the differential equation (1.1); $X^0(t) = S_t X^0$ is the solution of (1.1) with initial condition X^0 and $\gamma = \{X^0(t), 0 \leqslant t \leqslant T\}$ is the corresponding trajectory in $\mathbb{R}^d$.

Let $\Gamma = \{X \mid x_j = a\}$, $j \in \{1, \ldots, d\}$ fixed, be a given hyperplane in the space $\mathbb{R}^d$, ρ be a fixed parameter which will be suitably chosen later, $W_\rho(\gamma)$ be the ρ-neighborhood of γ, and $U_\rho(X^0)$ be the ρ-neighborhood of the point $X^0 \in \Gamma$ of the form

$$U_\rho(X^0) = \left\{ X \mid \sum_{i \neq j} (x_i - x_i^0)^2 < \rho^2, |x_j - a| < \rho \right\}$$

$F'(X)$ is the matrix with elements $F'_{i,k}(X) = \partial f_i(X)/\partial x_k$.

We shall make use of the linearized equation

$$\frac{dz}{dt} = F'(X^0(t))z \tag{1.2}$$

Let $\mathscr{L}(s,t)$ be the fundamental matrix of solutions of (1.2). For any T we define

$$C_1 = \sup_{0 \leqslant s,t \leqslant T} \|\mathscr{L}(s,t)\|$$

The norm of the matrix $\mathscr{L}$ is given by

$$\|\mathscr{L}\| = (\text{max eigenvalue of } \mathscr{L}\mathscr{L}^*)^{1/2}$$

C_2 is a constant such that

$$\sum_{i,l,k} \left| \frac{\partial^2 f_i}{\partial x_l \partial x_k} Y_l Z_k \right| \subseteq C_2 |Y| |Z| \qquad \text{for any} \quad Y, Z \in R^d$$

We shall use the symbol F_i'' for the matrix $\partial^2 f_i/\partial x_l \partial x_k$ $l,k = 1, \ldots, d$. The existence of C_2 can be derived from the form of $F(X)$. Further

$$C_3 = \inf_{x \in U_\rho(X^0)} |f_j(X)|, \qquad C_4 = \sup_i \sup_{x \in U_\rho(X^0)} |f_i(X)|$$

$$C_5' = \sup_{x \in U_\rho(X^0)} \|F'(X)\|, \qquad C_5 = \sup_{x \in W_\rho(\gamma)} \|F'(X)\|$$

We now state the main criterion for a rigorous study of the periodic orbits. Let Γ be the hyperplane chosen above and $X^0 \in \Gamma$, let T be the least positive time such that $S_T X^0 \in \Gamma$. Let us introduce the following notations:

$$\bar{X} = S_T X^0, \qquad Y = X - X^0, \qquad Y^0 = \bar{X} - X^0, \qquad \epsilon = |\bar{X} - X^0|$$

If $\mathscr{P}$ is the Poincaré map induced by the flow S_t on the hyperplane Γ, we have $\mathscr{P}X^0 = \bar{X}$. We expand the Poincaré map in the neighborhood $U_\rho(X^0) \cap \Gamma$:

$$Q(Y) = \mathscr{P}(X) - X^0, \qquad Q(Y) = Y^0 + LY + K(Y)$$

L is the derivative of the Poincaré map evaluated at $X = X^0$ and $K(Y)$, as we shall prove, satisfies the following condition:

A: There exists ρ_0 and $k_0 > 0$ such that for any $\rho < \rho_0$ and any $Y^1 = X^1 - X^0$, $Y^2 = X^2 - X^0$, $X^1 X^2 \in \Gamma$, $|Y^2| < \rho$, $|Y^2| < \rho$

$$|K(Y^1) - K(Y^2)| \leqslant k_0 \rho |Y^1 - Y^2|$$

Criterion. Let $|Y^0| = \epsilon$ and suppose that for some $\rho_1 < \rho_0$

$$\|(L-E)^{-1}\|(\epsilon/\rho_1 + k_0\rho_1) < 1 \tag{1.3}$$

Then there exists a unique fixed point of the Poincaré map in the ρ_1-neighborhood of the point X^0.

For the proof of this criterion see Ref. 1.

2. ESTIMATE OF THE NONLINEAR PART OF THE POINCARÉ MAP

In order to evaluate the constant k_0 appearing in the criterion we need some simple and useful lemmas.

Lemma 1. If $\rho_0 = 1/C_1^2 C_2 T$, $|X - X^0| < \rho_0$, then for $0 \leqslant t \leqslant T$

$$|S_t X - S_t X^0| \leqslant 2C_1 |X_1 - X_0| \tag{2.1}$$

Proof. We use the definition $X(t) = S_t X$, $X^0(t) = S_t X^0$

$$\begin{aligned}\frac{d}{dt}\left[X(t) - X^0(t)\right] &= F(X(t)) - F(X^0(t)) \\ &= F'(X^0(t))\left[X(t) - X^0(t)\right] \\ &\quad + \tfrac{1}{2}\left(F''(X(t) - X^0(t)), (X(t) - X^0(t))\right)\end{aligned} \tag{2.2}$$

We write the equation (2.2) in this way:

$$X(t) - X^0(t) = \mathscr{L}(0,t)(X - X^0)$$

$$+ \int_0^t ds\, \mathscr{L}(s,t)\frac{1}{2}\left(F''(X(s) - X^0(s)), (X(s) - X^0(s))\right)$$

$$|X(t) - X^0(t)| \leqslant C_1|X - X^0| + \frac{C_1 C_2}{2}\int_0^t ds\, |X(s) - X^0(s)|^2$$

Let $Z(t)$ be the solution of the integral equation

$$Z(t) = C_1|X - X^0| + \frac{C_1 C_2}{2}\int_0^t ds\, Z^2(s)$$

then

$$|X(t) - X^0(t)| \leqslant Z(t) = \frac{C_1|X - X^0|}{1 - (C_1^2 C_2 t/2)|X - X^0|} \leqslant 2C_1|X - X^0|$$

Lemma 2. If $|X^1 - X^0| < \rho_0$, $|X^2 - X^0| < \rho_0$, then for $0 \leqslant t \leqslant T$

$$|S_t X^1 - S_t X^2| \leqslant 8C_1|X^1 - X^2| \tag{2.3}$$

The proof is analogous to the proof of Lemma 1.

Now we are able to verify the condition A and to estimate k_0.

Proposition 1. If $\rho < \rho_0$ and $|X^1 - X^0| < \rho$, $|X^2 - X^0| < \rho$, $X^1, X^2 \in \Gamma$ then

$$|K(Y^1) - K(Y^2)| \leqslant k_0\rho|Y^1 - Y^2| \tag{2.4}$$

where $Y^1 = X^1 - X^0$, $Y^2 = X^2 - X^0$ and

$$k_0 = 16C_1^2\left(1 + \frac{C_4}{C_3}\right)\left[C_1C_2T + \frac{C_5}{C_3}\left(2 + \frac{C_4}{C_3}\right)\right]$$

Proof. Let T_1, T_2, T be such that $S_{T_1}X^1 \in \Gamma$, $S_{T_2}X^2 \in \Gamma$, $S_T X^0 = \overline{X} \in \Gamma$. We have to estimate the quantity

$$K(Y^1) - K(Y^2) = X^1(T_1) - X^2(T_2) - L(Y^1 - Y^2)$$

Using the following notation:

$$h^i(s) = \big(F''\big(X^i(s) - X^0(s)\big), \big(X^i(s) - X^0(s)\big)\big)$$

if we suppose for definiteness $T_2 < T_1$, we have

$$\begin{aligned} X^1(T_1) - X^2(T_2) &= X^1(T_1) - X^1(T) + X^1(T) \\ &\quad - X^2(T) + X^2(T) - X^2(T_2) \\ &= \mathscr{L}(0,T)(X^1 - X^2) + \frac{1}{2}\int_0^T \mathscr{L}(s,T)\big[h^1(s) - h^2(s)\big]\,ds \\ &\quad + \int_T^{T_2}\big[F\big(X^1(s)\big) - F\big(X^2(s)\big)\big]\,ds + F(\overline{X}^1)(T_1 - T_2) \end{aligned} \tag{2.5}$$

where $f_i(\overline{X}^1) = f_i(X^1(\tau_i))$ and $T_2 \leqslant \tau_i \leqslant T_1$, $i = 1, \ldots, d$.

If we observe now that $x_j^1(T_1) - x_j^2(T_2) = 0$ and if we put in (2.5) $F(\overline{X}^1) = F(\overline{X}) + F'(\hat{X})(\overline{X}^1 - \overline{X})$ we obtain the equality

$$\begin{aligned} &f_j(\overline{X})(T_1 - T_2) \\ &\quad = -\mathscr{L}_{jk}(0,T)\big(x_k^1 - x_k^2\big) - \frac{1}{2}\int_0^T ds\,\mathscr{L}_{je}(s,T)\big[h^1(s) - h^2(s)\big]_e \\ &\qquad - \int_T^{T_2}\big[f_j\big(X^2(s)\big) - f_j\big(X^2(s)\big)\big]\,ds - f_{je}'(\hat{X})\big(\overline{x}_e^1 - \overline{x}_e\big)(T_1 - T_2) \end{aligned} \tag{2.6}$$

We recall that (see Ref. 1):

$$L_{ik} = \mathscr{L}_{ik}(0, T) - f_i(\bar{X})\mathscr{L}_{jk}(0, T)/f_i(\bar{X})$$

and we replace in (2.5) the equality (2.6), we then find

$$\begin{aligned}|k(Y^1) - k(Y^2)| &\leqslant \left(1 + \frac{C_4}{C_3}\right)\left\{ \frac{1}{2}\int_0^T |\mathscr{L}(s,t)[h^1(s) - h^2(s)]|\, ds \right.\\ &\quad + \int_T^{T_2} |F(X^1(s)) - F(X^2(s))|\, ds\\ &\quad \left. + C_5'|X^1 - \bar{X}|\,|T_1 - T_2| \right\}\\ &\leqslant \left(1 + \frac{C_4}{C_3}\right)\left\{ 16C_1^3 C_2 \rho T |X^1 - X^2| + 8C_5' C_1 |T_2 - T| \right.\\ &\quad \left. \times |X^1 - X^2| + C_5'|\bar{X}^1 - \bar{X}|\,|T_1 - T_2| \right\}\end{aligned}$$

The proposition is proved if we observe finally that

$$|T_2 - T| \leqslant \frac{1}{C_3}|x_j^2(T_2) - x_j^2(T)| = \frac{1}{C_3}|x_j^0(T) - x_j^2(T)| \leqslant 2\frac{C_1}{C_3}\rho$$

analogously by Lemma 2

$$|T_1 - T_2| \leqslant 8\frac{C_1}{C_3}|X^1 - X^2|$$

$$|\bar{X}^1 - \bar{X}| = |X^1(\tau) - X^0(T)| \leqslant C_4|T_1 - T| + 2C_1\rho \leqslant 2C_1\rho\left(1 + \frac{C_4}{C_3}\right)$$

$$C_5' \leqslant C_5$$

3. THE INTEGRATION METHOD AND THE ESTIMATE OF THE ERROR

We integrate the differential equation (1.1) using the method of integrated approximations with step Δ

$$X_{(0)}(t) = X^0$$

$$\vdots$$

$$X_{(i)}(t) = X^0 + \int_0^t ds\, F(X_{(i-1)}(s)), \qquad 0 \leqslant t \leqslant \Delta, \quad i \leqslant m \tag{3.1}$$

The integration is done with m and Δ variable in order to see the dependence of the results on their choices.

The error of such a method at each step is of the order of

$$(C_5\Delta)^{m+1}(m+1)!$$

Let α be the error of the computer in a single integration step: i.e., $|X_{k+1} - RX_k| < \alpha$, where $X_0, X_1, \ldots$ is the pseudotrajectory and R denotes the application of the numerical method (3.1) to the point X_k.

If we look for the intersection with the hyperplane Γ of the pseudotrajectory, corresponding to the Poincaré map, we can reduce by a constant factor the integration step at every intersection of the pseudotrajectory with Γ and thus we find a point X^* and a period T.

The quantity ϵ used in the criterion can be evaluated as follows: let $n + \sum_i n_i$ be such that $X^* = X_{n+\sum_i n_i}$, n being the number of integrations with step Δ, n_i the number of integrations with step Δ_i; then

$$\epsilon = |\bar{X} - X^0| \leqslant |\bar{X} - S_{n\Delta+\sum_i n_i\Delta_i}X^0| + |S_{n\Delta+\sum_i n_i\Delta_i}X^0 - X^*| + |X^* - X^0| \tag{3.2}$$

Since for a periodic orbit the greatest contribution to ϵ is $|S_{n\Delta+\sum_i n_i\Delta_i}X^0 - X^*|$ we give the following:

Proposition 2. If $\alpha < 1/16C_1^3C_2Tn$ then

$$|X_k - S_{k\Delta}X^0| \leqslant 16kC_1\alpha$$

Proof. It follows from the numerical method used by us that [4]

$$|X_k - S_\Delta X_{k-1}| \leqslant (C_5\Delta)^{m+1}(m+1)! + \alpha < 2\alpha$$

then by Lemma 2 applied to the ρ_0-neighborhood $W_{\rho_0}(\gamma)$

$$|S_{j\Delta}X_k - S_{j\Delta}(S_\Delta X_{k-1}) \leqslant 16C_1\alpha$$

Let us write $|X_k - S_{k\Delta}X^0|$ in this way:

$$X_k - S_{k\Delta}X^0 = \sum_{j=0}^{k-1}\left[S_{j\Delta}X_{k-j} - S_{j\Delta}(S_\Delta X_{k-j-1})\right]$$

Thus

$$|X_k - S_{k\Delta}X^0| \leqslant \sum_{j=0}^{k-1} 16\alpha C_1|X_{k-j} - S_\Delta X_{k-j-1}| \leqslant 16C_1k\alpha$$

[4] We always choose m in such a way that α is bigger or of the same order of $(C_5\Delta)^{m+1}/(m+1)!$ at any integration step.

where we used the condition

$$\alpha \leqslant 1!\,16nC_1^3C_2T$$

in order to apply Lemma 2 at any k.

Using the interval arithmetic we estimate the quantity α and thus the total error ϵ.[7]

4. APPLICATION

We apply the approach described above to a periodic orbit appearing in the five-dimensional truncation of the Navier–Stokes equation.[6] We find numerically the fixed point of the Poincaré map using Newton's method which ensures convergence in few steps. The equations are

$$\begin{aligned} \dot{x} &= -2x + 4yz + 4uv \\ \dot{y} &= -9y + 3xz \\ \dot{z} &= -5z - 7xy + R \\ \dot{u} &= -5u - xv \\ \dot{v} &= -v - 3xu \\ R &= 25 \end{aligned} \tag{4.1}$$

We study the map induced on the plane $z = 3$ by the flow of solutions of (4.1). The coordinates of the numerical stable fixed point are

$$\begin{aligned} x_0 &= 0.46662024540 \\ y_0 &= 0.64215002423 \\ z_0 &= 3 \\ u_0 &= 0.687659513100 \\ v_0 &= -2.9799467541 \end{aligned} \tag{4.2}$$

The precision of the numerical fixed point is given by $|x^* - x^0| < 10^{-11}$. The values of the constants are $C_2 = 18$, $C_4 = C_3 = 8$, $C_5 = 32$, $C_5' = 23$, $k_0 = 5\ 10^5$, $T = 0.759$ $\|(L-E)^{-1}\| = 3$, $C_1 = 10$. C_1 is evaluated using the same method as in Ref. 1.

In order to find a bound for ϵ we use the interval arithmetic. We proceed as follows: once we have found the numerical fixed point we use another program where the numerical integration of the system of differential equation (4.1) is done in such a way that the m of (3.1) can be varied in order to satisfy at each integration step the condition on α used in Proposition 2. Furthermore to each numerical variable a numerical interval is associated in this program, the amplitude of the interval is equal to the error on the numerical variable due to the truncations of the computer and due to the propagation of the error generated by all the arithmetic operations made in order to find the value of the variable, and the center of the interval is equal to this value. We give to each variable x_0, y_0, z_0, u_0, v_0 in (4.2) an initial interval corresponding to the maximal precision of the computer, i.e., 10^{-17} for the Univac 1100 working in double precision.

Then all the arithmetic operations appearing in the integration method (3.1) are substituted by subroutines which compute the interval associated to the result of the arithmetic operation taking into account also the truncation and the finite precision of the computer. This allows us to evaluate the values of α and ϵ the value of which is less than 0.3×10^{-7}, and we checked also Proposition 2. We also checked that the greatest contribution to ϵ is given by this term because $|\bar{x} - S_{n\Delta + \Sigma_i u_i \Delta_i}|$ is of the order of 10^{-10} [see (3.2)]. Thus the criterion is satisfied with $\rho_1 = 0.510^{-6}$.

Theorem. In the 0.510^{-6} neighborhood of $(x_0, y_0, z_0, u_0, v_0)$ there exists a unique fixed point of the Poincaré map on the hyperplane $z = 3$.

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